M.Math. IInd year
First Semestral examination 2010
Number Theory — B.Sury
Answer any 6 questions including 3,6,8.

Q 1.
Let p be a prime. Show that the numerator of $\frac{n^p-n}{p}+\sum_{r=1}^{p-1}\frac{(-1)^r(1^r+2^r+\cdots+(n-1)^r)}{r}$ is a multiple of p.

Q 2.
Let p be a prime of the form $8k+3$. Assume that $4k+1$ is also prime. Prove that 2 is a primitive root modulo p.

OR

If p is a prime whose binary expansion looks like $100\cdots01$, prove that 3 is a primitive root mod p.

Q 3.
(i) Obtain the value of the periodic, simple, continued fraction $[1;\overline{2,3}]$.
(ii) Obtain the simple, continued fraction expansion of $\sqrt{a^2+1}$ for any natural number a.

Q 4.
Let $p=a^2+b^2$ be an odd prime with $a\equiv 1 \bmod 4$. Prove that $a+b$ is a quadratic residue mod p if and only if $(a+b)^2\equiv 1 \bmod 16$.

OR

(i) Spell out when the Jacobi symbol $(\frac{a}{N})$ is 1.
(ii) Let $p_1,p_2,\cdots,p_n$ be distinct, odd primes. Show the existence of a positive integer N such that the Jacobi symbol $(\frac{N}{p_1p_2\cdots p_n})=-1$.

OR

Let m be a product of primes of the form $4t+1$ and let n be an arbitrary integer. Prove that $y^2=x^3+(4n-1)^3-4m^2$ has no integral solutions.
Hint: Observe that any solution (x,y) satisfies $x\equiv 1 \bmod 4$; then rewrite the equality as $y^2+4m^2=x^3+(4n-1)^3$ and show that the right side must have a prime factor $\equiv 3 \bmod 4$ and derive a contradiction using quadratic reciprocity.

Q 5.
Prove that the quadratic form $7x^2+25xy+23y^2$ takes the same values as the quadratic form $x^2+xy+5y^2$ over integers.

Q 6.
Let $d \equiv 3 \bmod 4$ be square-free, positive integer. Let $\mathbf{O}$ be the ring $\mathbf{Z}[\frac{1+\sqrt{-d}}{2}]$. Determine the group of units of $\mathbf{O}$.

Q 7.
Let f be a multiplicative function. Consider the $n \times n$ matrix A where $a_{ij} = f(GCD(i,j))$. Show that $\det A = g(1)g(2)\cdots g(n)$ where $g(n) = \sum_{d|n} \mu(d) f(n/d)$.

OR

Prove that every even perfect number must be of the form $2^{p-1}(2^p - 1)$ with $2^p - 1$ prime.

OR

Prove:
(i) $\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\phi(d)}$.
(ii) $\mu(n) = \sum_{(k,n)=1} e^{2ik\pi/n}$.

Q 8.
Prove that for each $\epsilon > 0$, there exists a natural number n such that $\phi(n) < \epsilon n$. Deduce that $\pi(x) = o(x)$.

OR

Define $\theta(x) = \sum_{p \leq x; p\ prime} \log(p)$, $\psi(x) = \sum_{n \geq 1} \theta(x^{1/n})$ and $T(x) = \sum_{n \leq x} \log(n)$.
(i) Prove that $T(x) = \sum_{n \leq x} \psi(x/n)$.
(ii) Prove that $\lim_{x \to \infty}(\frac{\theta(x)}{x} - \frac{\psi(x)}{x}) = 0$.

Q 9.
Using Bertrand's postulate or otherwise, prove:
$n! = m^k$ does not have solutions for $m, n, k > 1$.

OR

Using Bertrand's postulate or otherwise, prove:
Every natural number can be written as a sum of finitely many distinct primes or one more than such a number.

Q 10.
Consider the sequence defined by $u_1 = 7, u_2 = 17$ and $u_{n+2} = 5u_{n+1} - 6u_n$ for $n > 0$. Determine explicitly a closed expression for the u_n's.

OR

Show that if n divides $2^n - 1$, then $n = 1$.